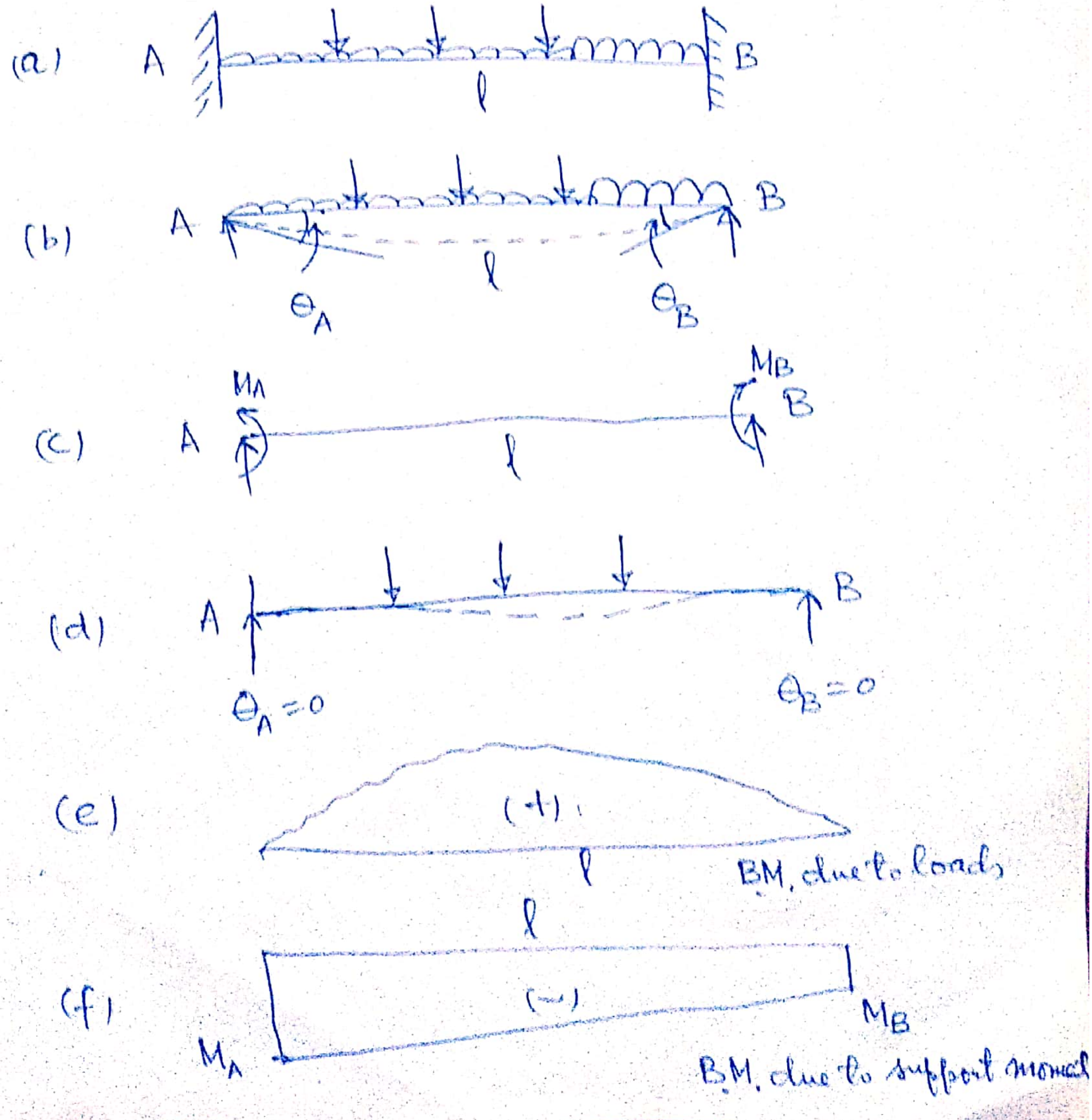
Fixed Beams

A beam which is built in at its two supports is called a Fixed Beam. A support, which is fixed, is constrained against rotation. The slope at the ends of a fixed beam is zero. Thus a beam AB (pls see figure) which is fixed at its ends may be treated as a simply supported beams subjected to end moment M_A & M_B such that the slopes at the two supports 'A' & 'B' are zero. This is only possible if the direction of the moments at the ends, M_A & M_B, is such that these oppose the rotation caused in the beam at the supports by the loads.



Thus the direction of support moment at A i.e; M_A needs to be aticlockwise and the direction of support moment at B i.e; M_B needs to be clockwise. The effect of these support moments is to cause a hogging bending moment throughout the span of the beam. The bending moment caused at support 'A' is M_A (hogging) and at support 'B' is M_B (hogging). Along the span of beam AB it is linear variation with M_A at 'A' and M_B at B. The bending moments caused in the beam AB are shown in figure (e) (due to loads) and figure (f) due to support moments.

Advantages of fixed beams;

(i) The beam is stiffer, stronger and more stable.

(ii) The deflections caused in the beam are very small as compared to those of a S.S.B under the same conditions.

Analysis of fixed beams:

Consider a beam AB fixed at the supports 'A' & 'B' (refer figure 2). The beam may be analysed by treating the beam as a combination of two simply supported beams: One carrying the loads and the other subjected to support moment. (Please refer figure-2)

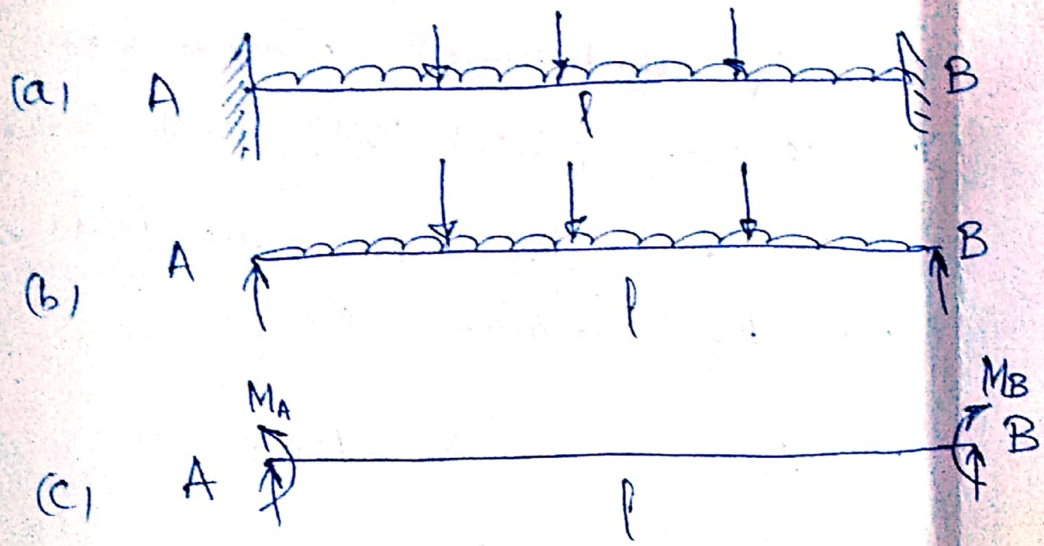
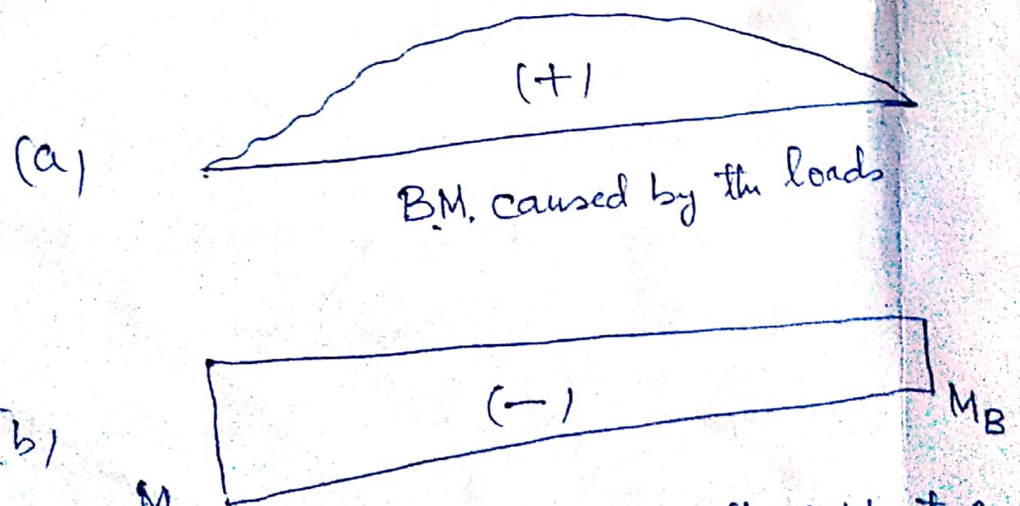


figure-2

The bending moments caused are shown in figure-3



BM. caused by the loads

BM. caused by the support moment

figure-3

As per Mohr's 1st theorem of Moment Area

$$\theta_A - \theta_B = \text{Area of BMD}/EI$$

$$0 = \text{Area of BMD}/EI$$

If 'A_s' is the area of B.M.D. caused by loads (shown in figure-3 (a)) & 'A_f' is the area of BMD caused by support moments (shown in figure - 3(b)).

Then Area of total B.M.D. on the beam AB is $A_s - A_f$ since A_s is sagging (+ve) and A_f is hogging ~~negative~~ (−ve).

$$\therefore \quad 0 = (A_s - A_f)/EI$$

$$\text{or} \quad A_f = A_s$$

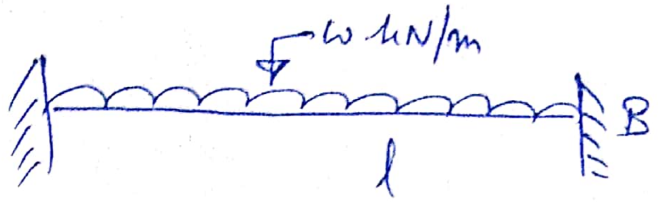
Thus Area of bending moment diagram caused by support moments is equal to the area of bending moment diagram caused by the loads.

Also as per Mohr's second theorem of Moment Area, if the vertical reference axis is drawn passing through support 'A' then the intercept of tangents drawn to the elastic curve of the beam AB at 'A' & 'B' on the reference axis is zero. Thus if $\bar{x}$ is the centroidal distance of bending moment diagram caused by loads from support 'A' and $\bar{x}'$ is the centroidal distance of bending moment diagram caused by support moments M_A & M_B from 'A', then

$$\left(A_s\bar{x} - A_f\bar{x}'\right)/EI = 0$$

or $$A_f\bar{x}' = A_f\bar{x}$$

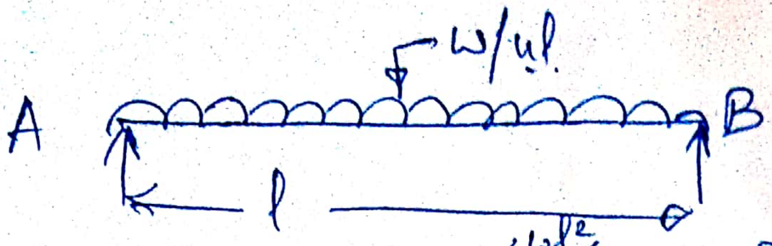
Example 1: A beam 'AB' fixed at the supports 'A' & 'B' carries a udl w kN/m over the entire span. Analyse the beam, Draw detailed BM diagram.

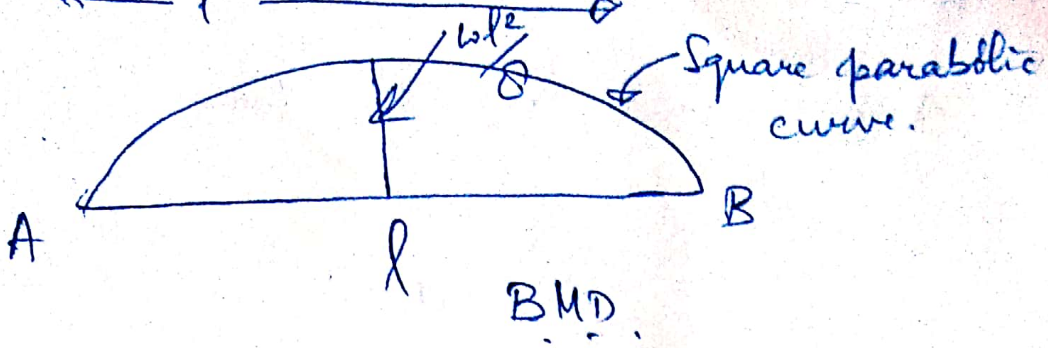


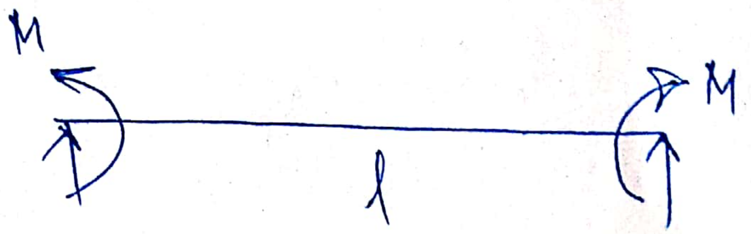
Let the support moments be M_A & M_B.

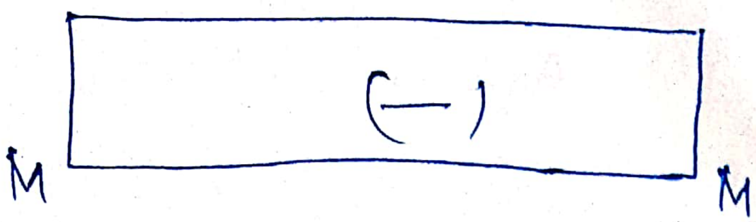
The beam can be treated as two simply supported beams of span 'l', one carrying the u.d.l. and the other subjected to support moments M_A & M_B. Here

As the load is symmetrically placed, $M_A = M_B = M$ (say).









$$A_s = \frac{2}{3} \times l \times \frac{wl^2}{8} = \frac{wl^3}{12}$$

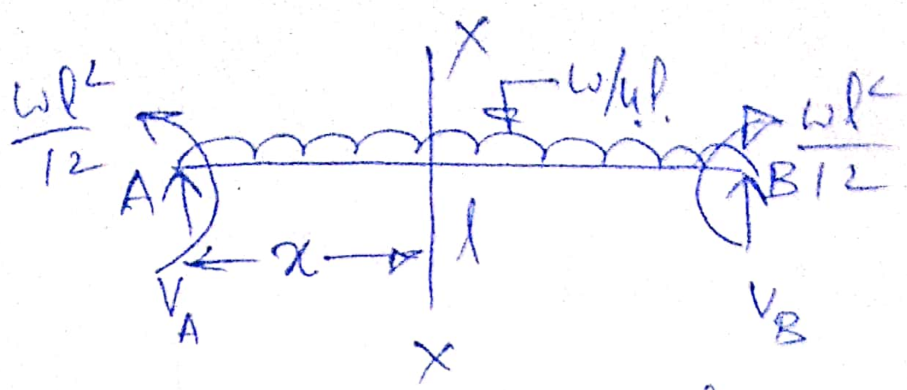
$$A_f = M \times l = Ml$$

$$A_f = A_s$$

$$Ml = \frac{wl^2}{12}$$

$$\Rightarrow M = \frac{wl^2}{12}$$

Now the beam can be shown as:



Vertical reaction $V_A = V_B = \frac{wl}{2}$

BM at X–X $\quad M_x = \frac{wl}{2}.x - \frac{wx^2}{2} - \frac{wl^2}{12}$

(This is a square parabolic curve)

At A, $x = 0$ $\quad M_A = \cancel{\frac{wl}{2}.x} - \frac{wl^2}{12}$

At B, $x = l$ $\quad M_B = \frac{wl}{2}.l - \frac{wl^2}{2} - \frac{wl^2}{12} = \frac{-wl^2}{12}$

For points of contraflexure, $\cancel{\frac{\partial M_x}{\partial x} = 0}$ $M_x = 0$

$\cancel{\frac{wl}{2} - wx = 0}$ $\quad \frac{wl}{2}.x - \frac{wx^2}{2} - \frac{wl^2}{12} = 0$

or $\quad -\frac{w}{2}\left[x^2 - lx + \frac{l^2}{6}\right] = 0$

or $\quad x^2 - lx + l^2 = 0$ $\quad$ Since $\frac{w}{2} \neq 0$

$\Rightarrow \quad x = \frac{+l \pm \sqrt{l^2 - 4.1.\frac{l^2}{6}}}{2} = \frac{+l \pm \sqrt{\frac{1}{3}}.l}{2}$

$$x = \frac{l \pm l\sqrt{2/3}}{2} = \frac{l\left(1 \pm \sqrt{2/3}\right)}{2}$$

$$= \frac{l\,(1 \pm 0.577)}{2} = 0.2115l,\ 0.7885l$$

$$\approx 0.2l \ \& \ 0.8l$$

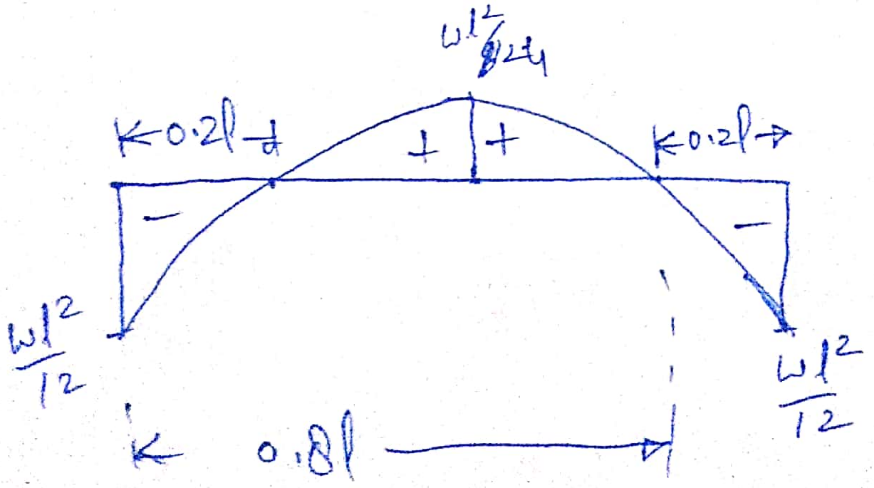
The B.M. diagram can be drawn as;

B.M. at mid span (i.e, at $x = l/2$)

$$M_{l/2} = \frac{wl}{2}\cdot\frac{l}{2} - \frac{w}{2}\left(\frac{l}{2}\right)^2 - \frac{wl^2}{12}$$

$$= \frac{wl^2}{4} - \frac{wl^2}{8} - \frac{wl^2}{12} = \frac{wl^2}{24}\left[6-3-2\right)$$

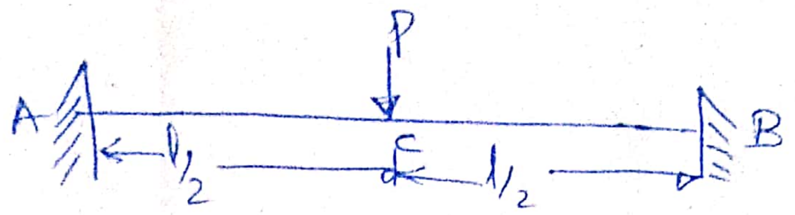
$$= +\frac{wl^2}{24}$$



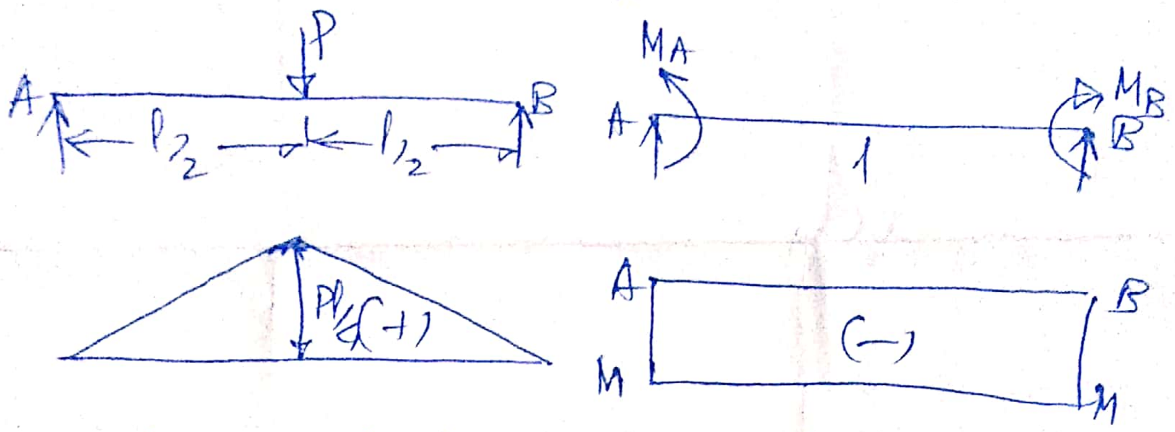
Example 2 :

A fixed beam AB of span 'l' carries a concentrated load 'P' at mid span. Analyse the beam. Draw detailed BMD.

Sol.



The given beam shall be treated as two S.S. beams, one containing the load, other subjected to support moments as shown;

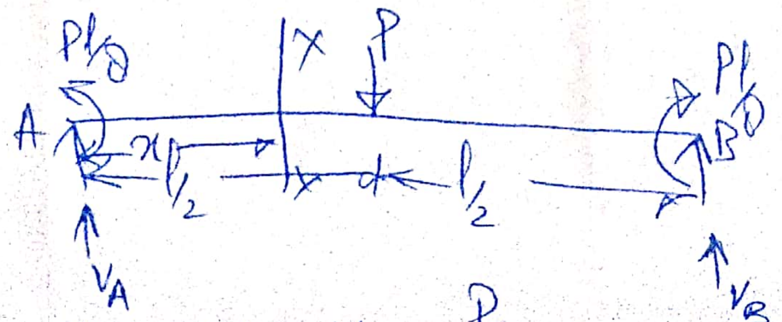


Since the load is placed symmetrically, $M_A = M_B = M$ (say)

$$A_s = \frac{1}{2} \times l \times \frac{Pl}{4} = \frac{Pl^2}{8}; \qquad A_f = M.l$$

$$A_f = A_s \Rightarrow Ml = \frac{Pl^2}{8} \Rightarrow M = \frac{Pl}{8}$$

Now the beam can be shown as:



The vertical reactions $V_A = V_B = \frac{P}{2}$

BM. at x-x

$$M_x = V_A \cdot x - \frac{Pl}{8} = \frac{P}{2}x - \frac{Pl}{8}$$

This is equation of a st. line.

BM. at A $(x=0)$ $= -\frac{Pl}{8}$

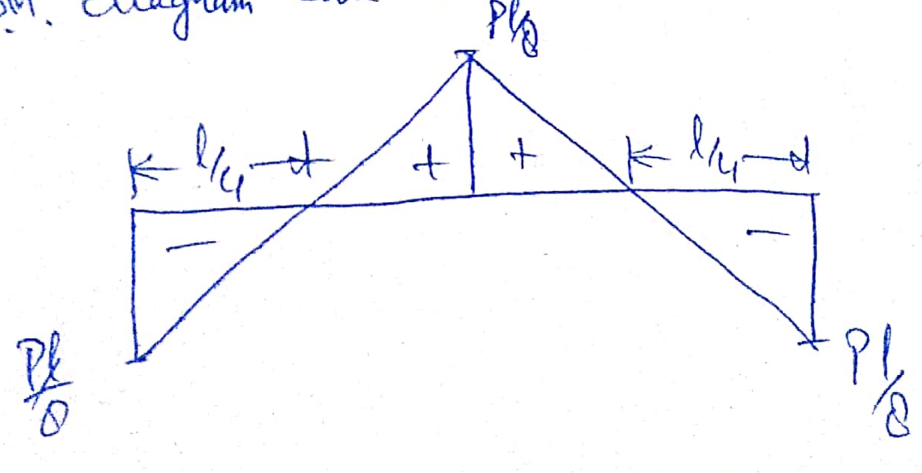
BM at C $(x = l/2)$ $= \frac{P}{2} \cdot \frac{l}{2} - \frac{Pl}{8} = \frac{Pl}{4} - \frac{Pl}{8} = +\frac{Pl}{8}$

For Points of contraflexure,

$$\frac{P}{2}x - \frac{Pl}{8} = 0 \Rightarrow \frac{P}{2}\left(x - \frac{l}{4}\right) = 0$$

$$\frac{P}{2} \neq 0 \quad x - \frac{l}{4} = 0 \quad x = \frac{l}{4}$$

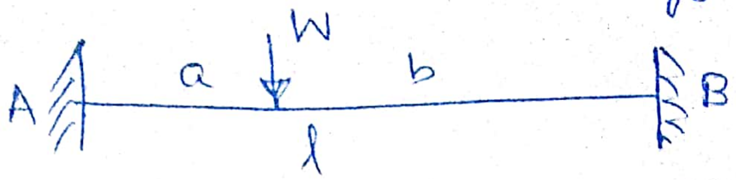
The BM. diagram can be drawn as



Symmetrical due to symmetry of load.

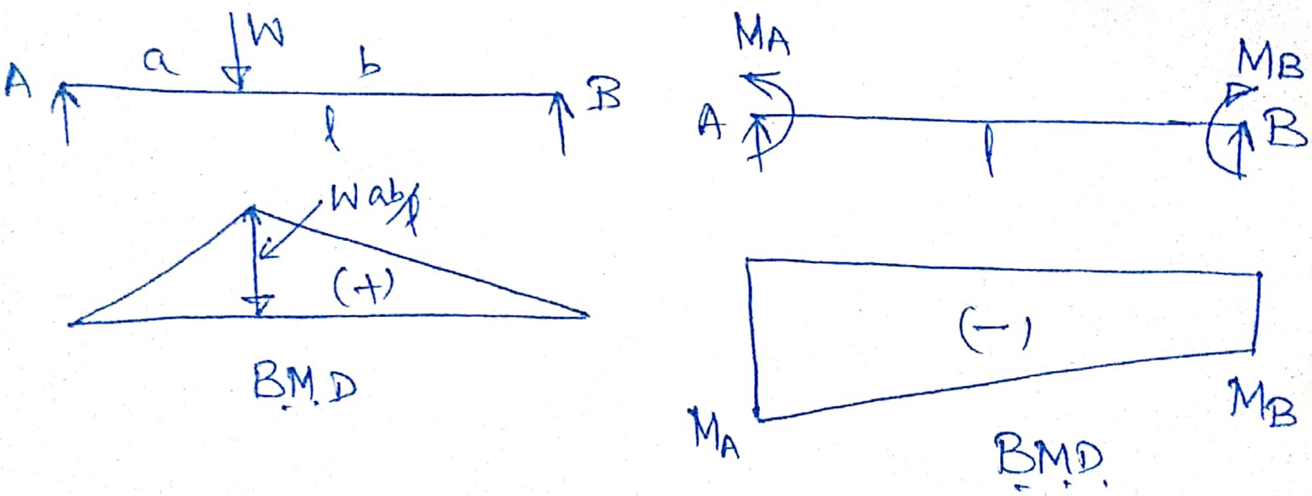
Example 3

A beam 'AB' fixed at both supports 'A' & 'B' carries a concentrated load 'W' as shown. Analyze the beam.



Solution:

The beam is treated as two simply supported beams, one carrying the load, the other subjected to support moments as shown.



Since the load is not symmetrically placed, $M_A \neq M_B$.

Area of BMD caused by the load

$$A_s = \frac{1}{2} \times l \times \frac{Wab}{l} = \frac{Wab}{2}$$

Area of BMD caused by support moments,

$$A_f = \frac{M_A + M_B}{2} \times l = \frac{l}{2}\left(M_A + M_B\right)$$

Moment of BMD caused by load about 'A'

$$A_s \bar{x} = \left[\frac{1}{2} \times \frac{Wab}{l} \times a \times \frac{2a}{3}\right] + \left[\frac{1}{2} \cdot \frac{Wab}{l} \times b \times \left(a + \frac{b}{3}\right)\right]$$

$$= \frac{Wab}{6}(2l - b) = \frac{Wab}{6}(2a + b)$$

Now $A_f = A_s$

$$\left[M_A + M_B\right] \frac{l}{2} = \frac{Wab}{2}$$

$$M_A + M_B = \frac{Wab}{l} \quad \text{— (I)}$$

Also $A_f \bar{x}' = A_s \bar{x}$

$A_f x' =$ Moment of BMD caused by support moment about 'A'

$$= \frac{l^2}{6} (M_A + 2M_B)$$

$$\therefore \quad \frac{l^2}{6} (M_A + 2M_B) = \frac{Wab}{6} (2a+b)$$

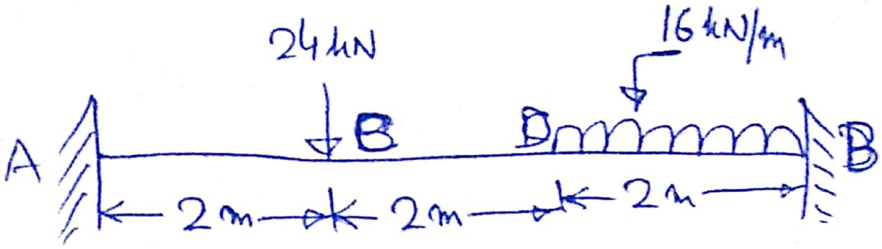
or $\quad M_A + 2M_B = \frac{Wab}{l^2}(2a+b) \quad \text{— (II)}$

Solving (I) & (II)

$$M_A = \frac{Wab^2}{l^2} \quad \& \quad M_B = \frac{Wa^2b}{l^2}$$

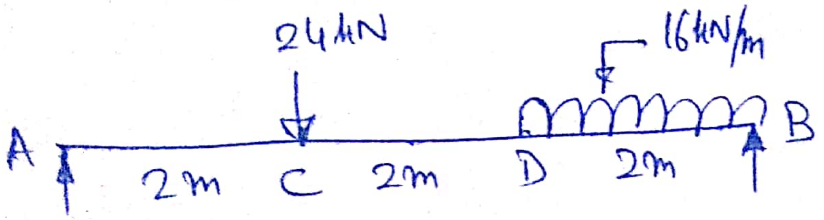
Example 4

Analyse the fixed beam AB (as shown in figure). Determine the support moments. Draw complete B.M. diagram.

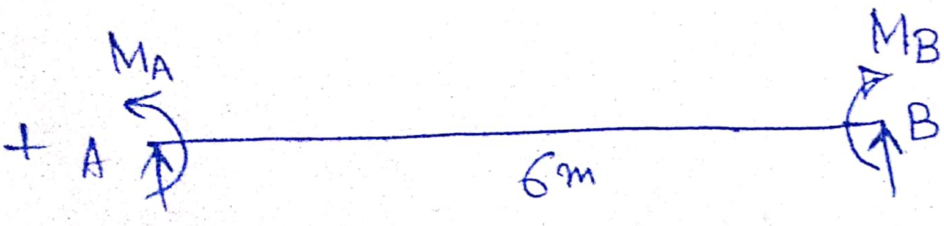


Solution:

The beam can be treated as two simply supported beams, one containing the loads and the other subjected to support moments as shown below:

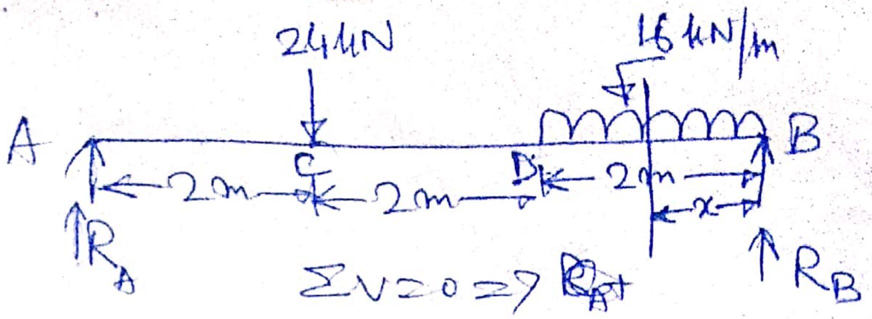


(a)



(b)

Drawing B.M. diagrams:



$\Sigma V = 0 \Rightarrow$

$R_A + R_B = 16 \times 2 + 24 = 56$ kN

$\Sigma M = 0$ @ A

$$R_B \times 6 = 24 \times 2 + 16 \times 2 \times 5$$

$$R_B = \frac{48 + 160}{6} = 34.666 \text{ kN}$$

$$R_A = 56 - 34.666 = 21.333 \text{ kN}$$

B.M. at C = $R_A \times 2 = 21.333 \times 2 = 42.666$ kN m

B.M. at D = $R_A \times 4 - 24 \times 2 = 21.333 \times 4 - 48$

$= 37.332$ kNm

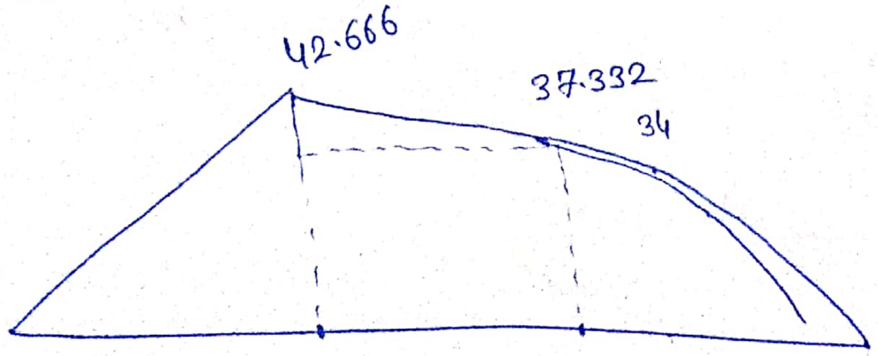
Also B.M. at D = $R_B \times 2 - 16 \times 2 \times \frac{2}{2}$

$= 34.666 \times 2 - 32 = 33.332$ kNm

B.M. at any section x–x at 'x' from B

$$M_x = R_B \cdot x - 16 \cdot x \cdot \frac{x}{2} = 34.666x - 8x^2$$

The B.M.D. can be drawn as



Area of BMD

$$A_s = \left(\frac{1}{2}\times 2\times 42.666\right) + \left(2\times 37.332\right) + \frac{1}{2}\times 2\times\left(42.666 - 37.332\right)$$

$$+ \int_0^2 \left(34.666x - 8x^2\right) dx$$

$$= 42.666 + 74.664 + 5.334 + \left[17.333x^2 - \frac{8x^3}{3}\right]_0^2$$

$$= 122.664 + 48 = 170.664$$

Moment of Area of BMD about B,

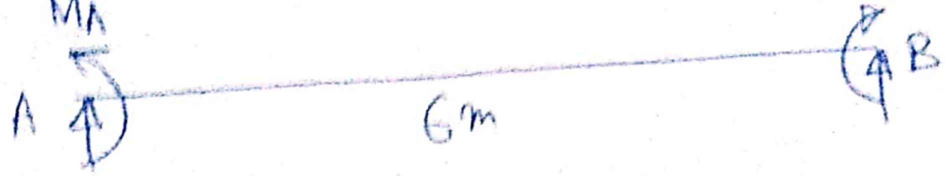
$$A_s\bar{x} = \left[42.666\times\left(4+\frac{2}{3}\right)\right] + \left[74.664\times(2+1)\right] + 5.334\left[2+\frac{2}{3}\times 2\right]$$

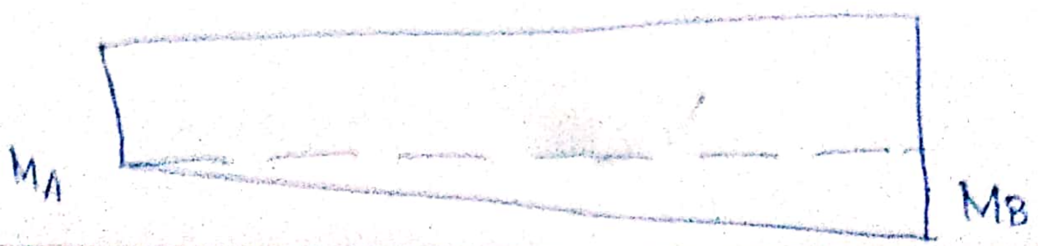
$$+ \int_0^2 \left(24.666x^2 - 8x^3\right) dx$$

$$= 199.08 + 224 + 17.78 + \left[8.222x^3 - 2x^4\right]_0^2$$

$$= 440.86 + 33.776 = 474.636$$

BMD of the Simply Supported beam with support moment.





Area of B.M. diagram

$$A_f = (M_A + M_B)\frac{6}{2} = 3(M_A + M_B)$$

Moment of area of B.M.D about B

$$A_f \bar{x}' = (M_A \times 6 \times 3) + \frac{1}{2}(M_B - M_A) \times 6 \times \frac{6}{3}$$

$$= 18M_A + 6M_B - 6M_A = 6(2M_A + M_B)$$

Now $\quad A_f = A_s$

$$3(M_A + M_B) = 170.664$$

$$M_A + M_B = 56.888 \quad \text{— (I)}$$

Also $\quad A_f \bar{x}' = A_s \bar{x}$

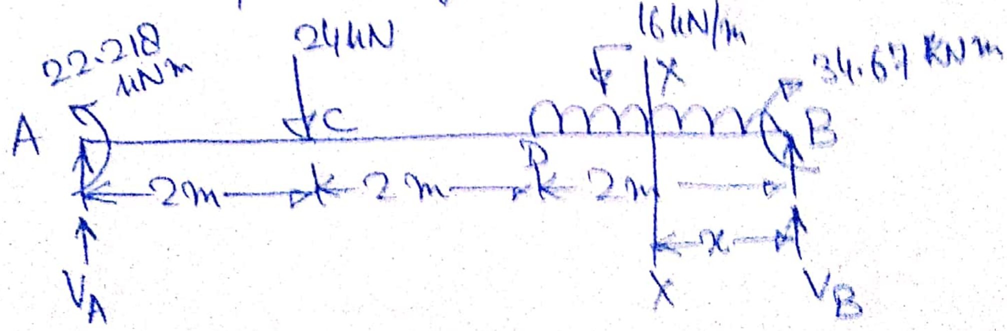
$$6(2M_A + M_B) = 474.636$$

$$2M_A + M_B = 79.106 \quad \text{— (II)}$$

Solving (I) & (II), we get

$$M_A = 22.218 \text{ kNm} \quad \& \quad M_B = 34.67 \text{ kNm}$$

Now the beam can be represented by



$$V_A + V_B = 24 + 16 \times 2 = 56 \text{ kN}$$

$\Sigma M @ B = 0$

$$V_A \times 6 + 34.67 = (24\times4) + \left(2\times16\times\frac{2}{2}\right) + 22.218$$

$$\Rightarrow V_A = \frac{96 + 32 + 22.218 - 34.67}{6} = 19.258 \text{ kN}$$

$$V_B = 56 - 19.258 = 36.742 \text{ kN}$$

BM. at A $= -22.218$ kNm

BM. at C $= -22.218 + V_A \times 2 = -22.218 + 19.258\times2$

$= +16.3$ kNm

BM. at D $= -22.218 + V_A\times4 - 24\times2$

$= -22.218 + 19.258\times4 - 48$

$= +6.814$ kNm

Also BM. at D $= -34.67 + V_B\times2 - 16\times2\times\frac{2}{2}$

$= -34.67 + 36.742\times2 - 32$

$= +6.814$ kNm

BM. at B $= -34.67$ kNm

BM. at X-X in DB at 'x' from B

$$M_x = +V_B x - 34.67 - 16.x.\frac{x}{2}$$

$$= 36.742x - 34.67 - 8x^2$$

for Max BM. in DB, $\frac{\partial M_x}{\partial x} = 0$

$\Rightarrow 36.742 - 16x = 0$

$\Rightarrow x = 2.296$ m [Max value of $x = 2$ m]

For point of contraflexure in DB

$M_x = 0$

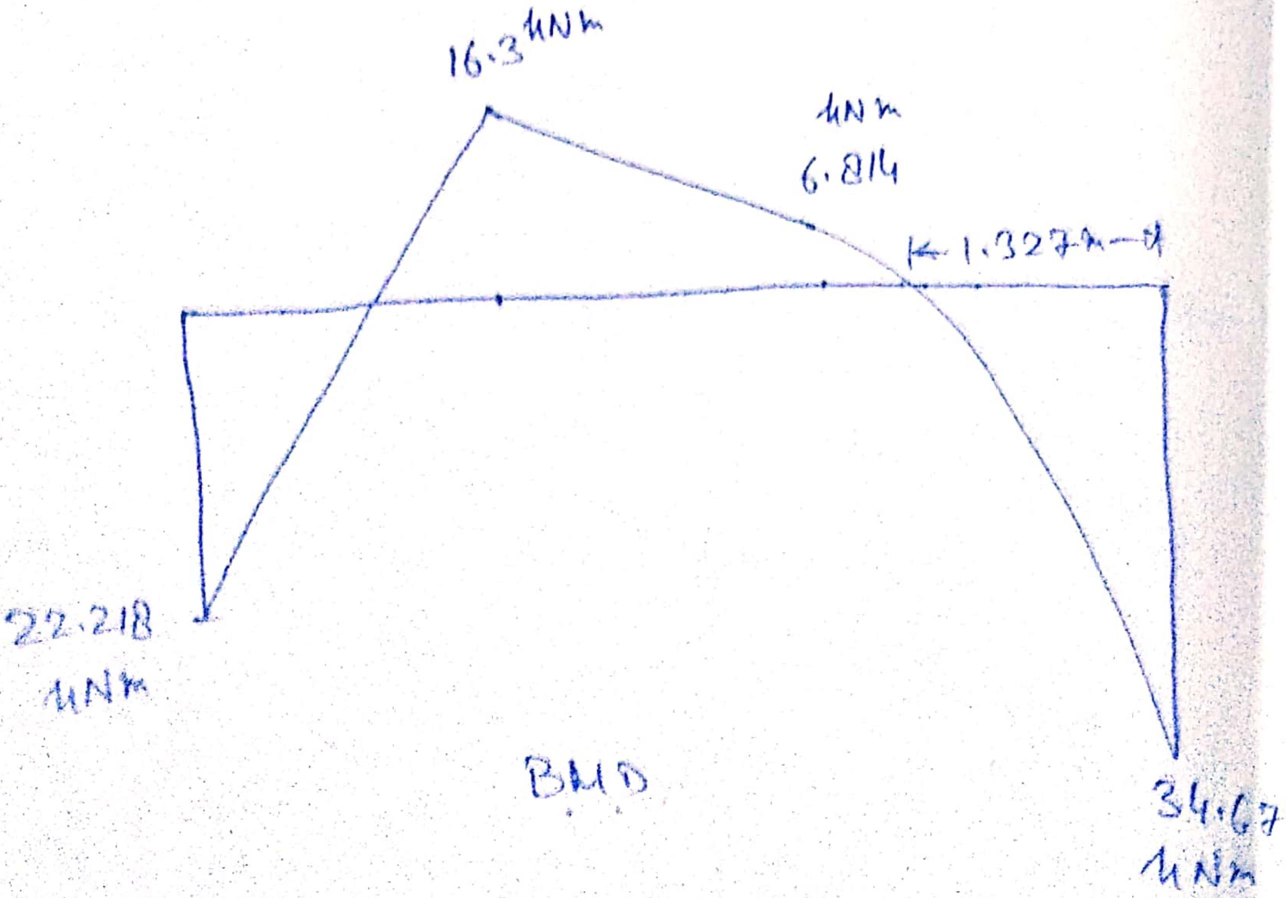
$\Rightarrow 36.742x - 34.67 - 8x^2 = 0$

$x = 1.327\,m;\ 3.264\,m$

But $x \not> 2$

Hence $x = 1.327\,m$

BMD is drawn as:



BMD